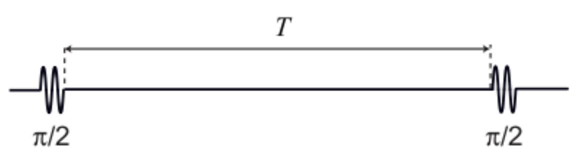
Physics 566 - Quantum Optics I
Problem Set 5 – Solutions

Problem 1: Measuring T_2 times via a Ramsey interferometer

(a) The standard Ramsey "separated zone" pulse sequence provides a method for measuring the coherence time in a quantum superposition between two orthogonal states $\{|0\rangle, |1\rangle\}$.



Given the qubit initially in the state $|1\rangle$. The probability to find the qubit in $|0\rangle$ after the pulse sequence was derived in lecture

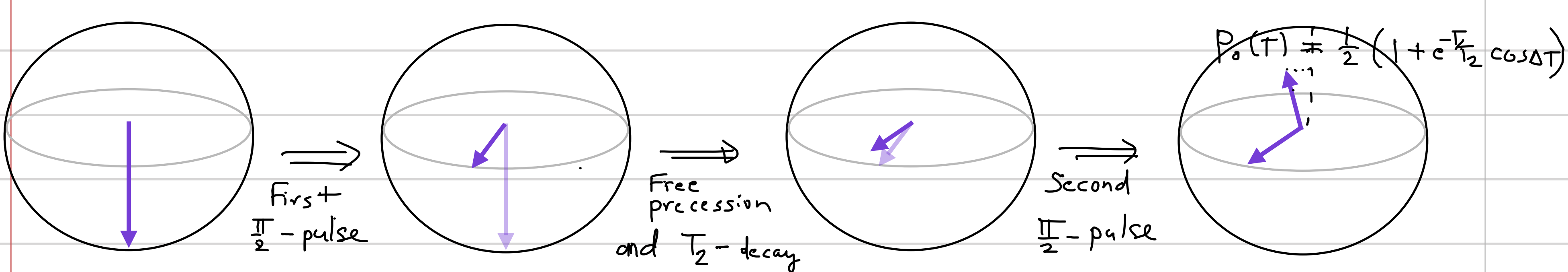
$$P_0(T) = \langle 0| e^{-i\frac{\pi}{4}\hat{\sigma}_x} \hat{\rho}(T) e^{i\frac{\pi}{4}\hat{\sigma}_x} |0\rangle = \left(\frac{\langle 0| - i\langle 1|}{\sqrt{2}} \right) \hat{\rho}(T) \left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}} \right)$$

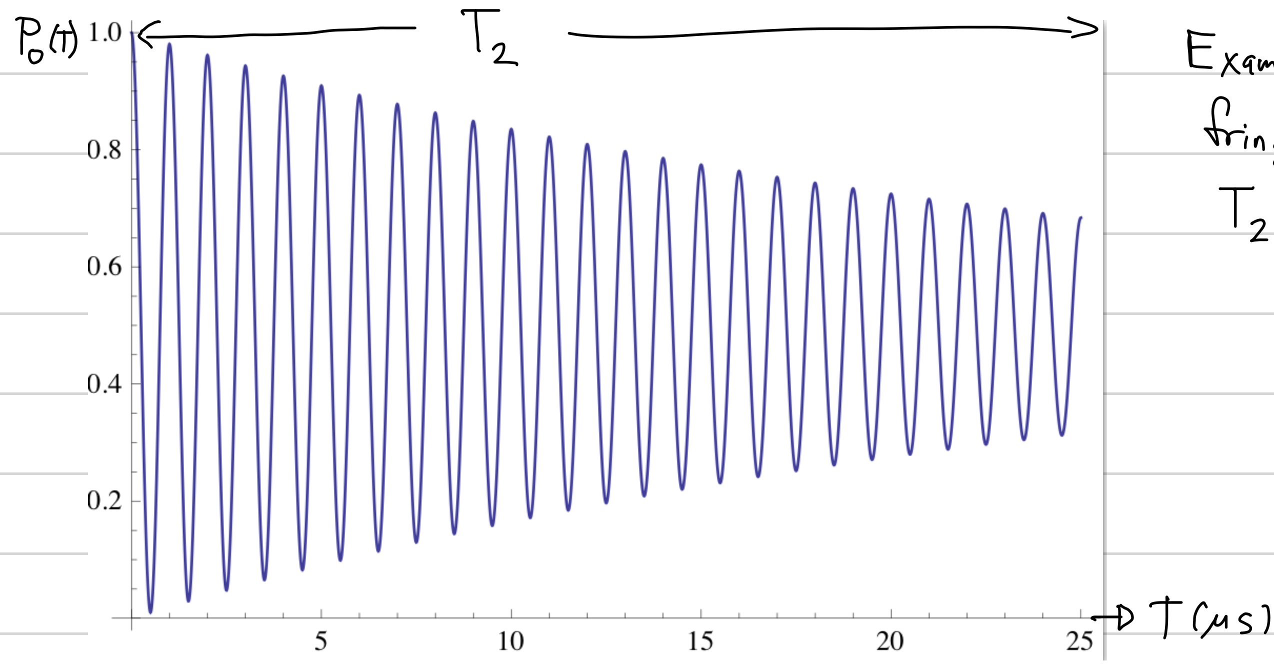
$$= \frac{1}{2}\left(\rho_{00}(T) + \rho_{11}(T) + i\left(\rho_{01}(T) - \rho_{10}(T)\right)\right) = \frac{1}{2}\left(1 + 2\,\mathrm{Im}\left(\rho_{10}(T)\right)\right)$$

Aside: $$\rho_{10}(T) = \underbrace{e^{-T/T_2}}_{T_2\text{-decay}} \underbrace{\langle 1|}_{\langle \downarrow_z|} e^{+i\frac{\Delta T}{2}\hat{\sigma}_z} \underbrace{e^{-i\frac{\pi}{4}\hat{\sigma}_x} \hat{\rho}(0)\, e^{i\frac{\pi}{4}\hat{\sigma}_x}}_{|\uparrow_y\rangle\langle\uparrow_y|} e^{-i\frac{\Delta T}{2}\hat{\sigma}_z} \underbrace{|0\rangle}_{|\uparrow_z\rangle} = e^{-T/T_2} \langle \downarrow_z| e^{\frac{i\Delta T}{2}\hat{\sigma}_z} |\uparrow_y\rangle\langle\uparrow_y| e^{-i\frac{\Delta T}{2}\hat{\sigma}_z} |\uparrow_z\rangle$$

$$\Rightarrow \rho_{10}(T) = \frac{i}{2}\, e^{-T/T_2}\, e^{-i\Delta T}$$

$$\Rightarrow \boxed{P_0(T) = \frac{1}{2}\left(1 + e^{-T/T_2}\cos(\Delta T)\right)}$$



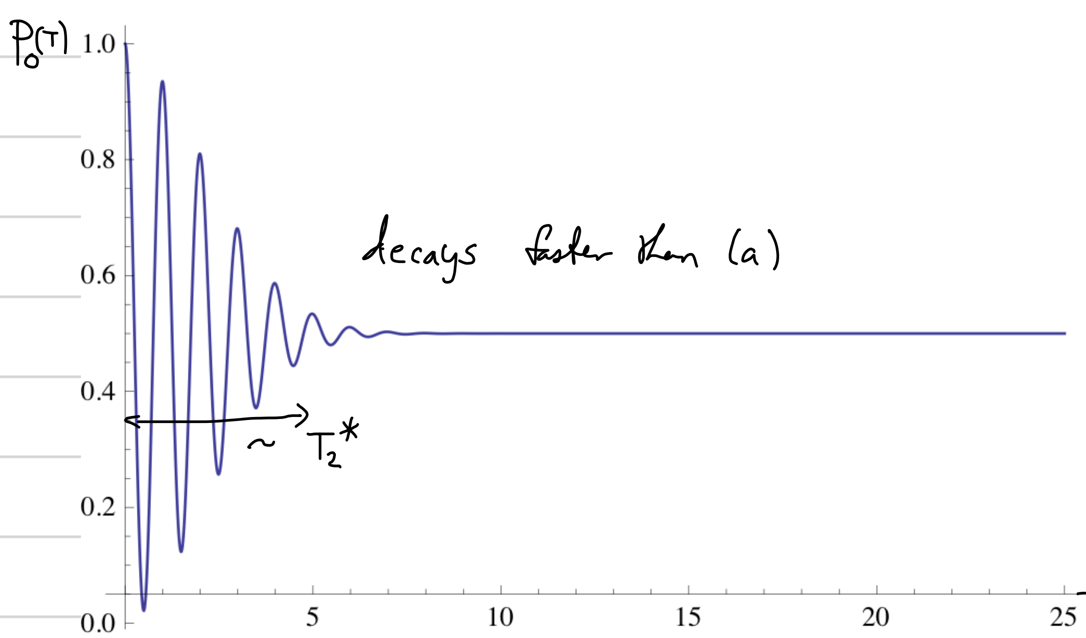


Example of decay Ramsey fringes for $\Delta/2\pi = 1$ MHz, $T_2 = 25\,\mu s$.

(b) Now we add inhomogeneous broadening – a distribution of detunings $p(\Delta) = \dfrac{e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}}}{\sqrt{2\pi\delta^2}}$

$$\Rightarrow P_0(T) = \int d\Delta\, p(\Delta)\, P_0(T,\Delta) = \int d\Delta\, p(\Delta) \left(\frac{1 + e^{-T/T_2} \cos(\Delta T)}{2} \right)$$

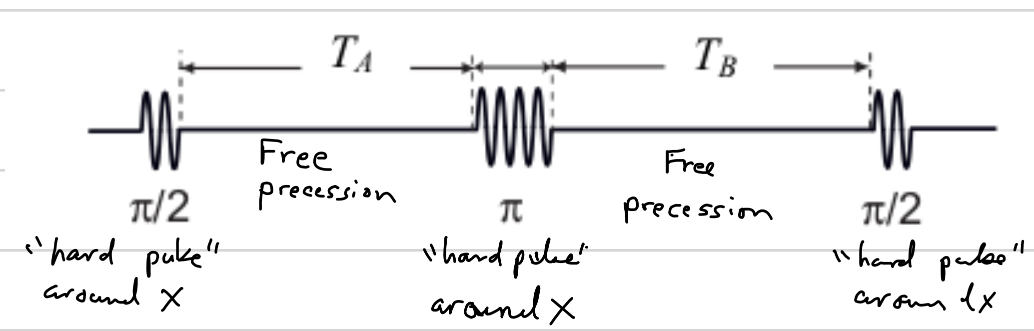
$$P_0(T) = \tfrac{1}{2}\left(1 + e^{-T/T_2}\, e^{-\frac{T^2}{2T_2^{*2}}} \cos(\Delta_0 T)\right) \qquad \text{where} \qquad T_2^* = \frac{1}{\delta}$$



$T\,(\mu s)$

Ramsey Two-pulse sequence with same mean detuning but inhomogeneously broadened, with a $T_2^* = 5\,\mu s < T_2 = 25\,\mu s$

(c) The Hahn spin echo sequence removes the inhomogenous brodening in Δ to first order.



The Spin-echo sequence is isomorphic to a Mach-Zender interferometer, as we have seen. For a given detuning, and without decay, the probability of transitioning $|1\rangle \to |0\rangle$

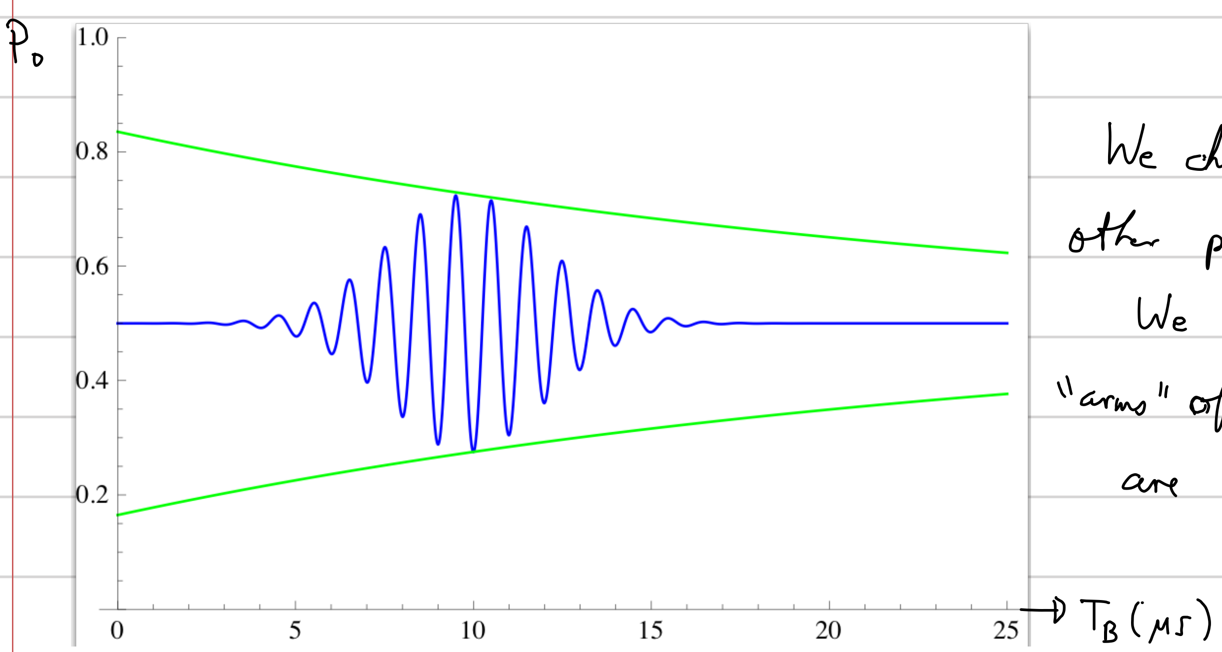
$$P_0(T_A, T_B, \Delta) = |\langle 0| e^{-\frac{i\Delta}{2}(T_A - T_B)\hat{\sigma}_y} |1\rangle|^2 = \sin^2\left(\frac{\Delta}{2}(T_A - T_B)\right) = \frac{1}{2}\left(1 - \cos(\Delta(T_A - T_B))\right)$$

Including homogeneous T_2 decay: $P_0(T, \Delta, T_2) = \frac{1}{2}\left[1 - \underbrace{e^{-\frac{T_A+T_B}{T_2}}}_{\text{decay during } T_A \text{ \& } T} \cos\{\Delta(T_A - T_B)\}\right]$

Including both T_2 and inhomogeneous T_2^*:

$$P_0(T_A, T_B, \Delta_0, T_2, T_2^*) = \int d\Delta \; p(\Delta, T_2^*, \Delta_0)\, P_0(T_A, T_B, \Delta, T_2)$$

$$\Rightarrow \boxed{P_0(T_A, T_B, \Delta_0, T_2, T_2^*) = \frac{1}{2}\left[1 - e^{\frac{-(T_A+T_B)}{T_2}} e^{-\frac{1}{2}\left(\frac{T_A - T_B}{T_2^*}\right)^2} \cos\{\Delta_0(T_A - T_B)\}\right]}$$



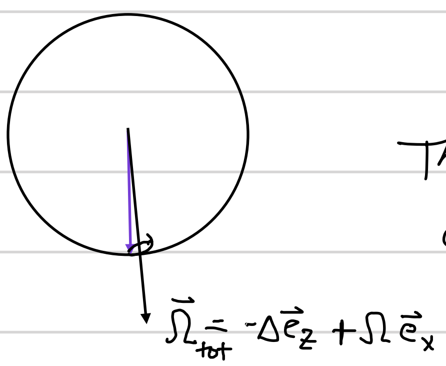
We chose here $T_A = 10\,\mu s$ – all other parameters are as in part (b). We see the "echo" we the two "arms" of the Ramsey interferometer are near equal $T_A \sim T_B$

The Hahn echo allows us to measure the true decay time T_2 (shown here as the green exponential decay curves), removing the dephasing due to inhomogeneity.

Problem 2: Adiabatic Rapid Passage

Using adiabatic evolution, we can robustly transfer population from ground to excited state without perfect knowledge of the parameters of the Hamiltonian

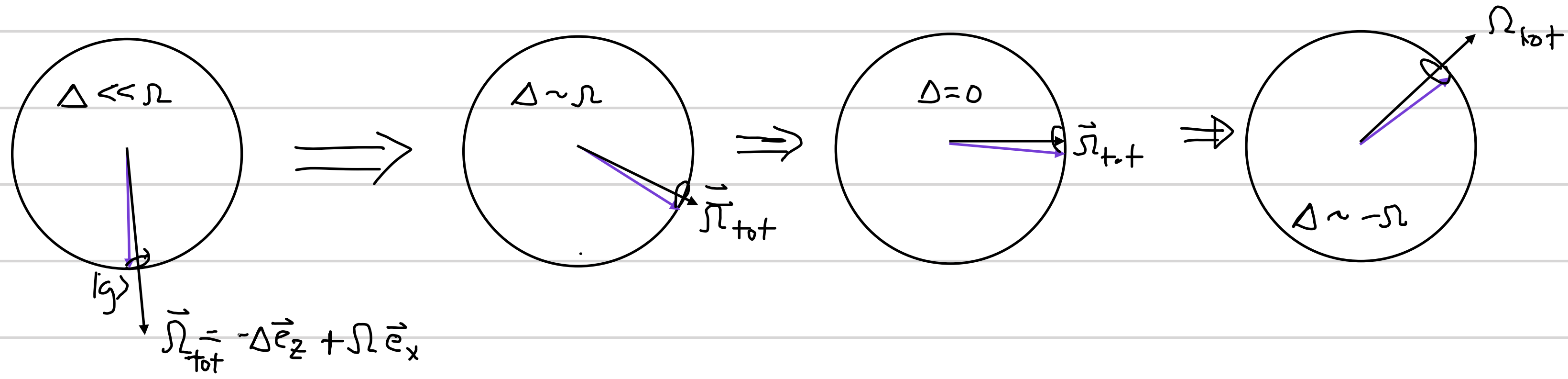
(a) Consider atoms initially in the ground state.
We apply our laser field that couples $|g\rangle$ and $|e\rangle$ at a detuning well below resonance

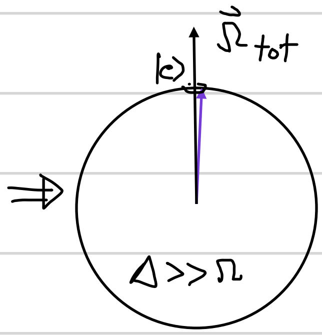


$|\Delta| >> \Omega$.

The Bloch vector will precess around the torque vector $\vec{\Omega}_{eff}$ close to the south pole.

Now suppose we sweep Δ through resonance slowly compared to $\sqrt{\Omega^2+\Delta^2}$. The spin will then "adiabatically follow" the direction of the torque vector regardless of the true value of Δ. The slowest rate is for $\Delta = 0 \Rightarrow$ we require the time scale over which we change $\vec{\Omega}$ to be slow compared to Ω. However, we must accomplish the transfer $|g\rangle \to |e\rangle$ much faster than the rate of spontaneous emission Γ. This kind of transfer is known as "adiabatic rapid passage" We must be adiabatic with respect to the coherent dynamics characterized by + frequency Ω, but rapid with respect to the decay process that occurs at rate Γ.





As long as the Bloch vector rapidly rotates around the torque vector, we can adiabatically rotate $|g\rangle \to |e\rangle$.

(b) Quantitatively, we can turn to the adiabatic theorem of quantum mechanics.

Our time-dependent Hamiltonian in the rotating frame is

$$\hat{H}_{RF}(t) = \hbar \vec{\Omega}_{tot}(t) \cdot \frac{\vec{\sigma}}{2}, \quad \vec{\Omega}_{tot}(t) = -\Delta(t)\vec{e}_z + \Omega\vec{e}_x$$

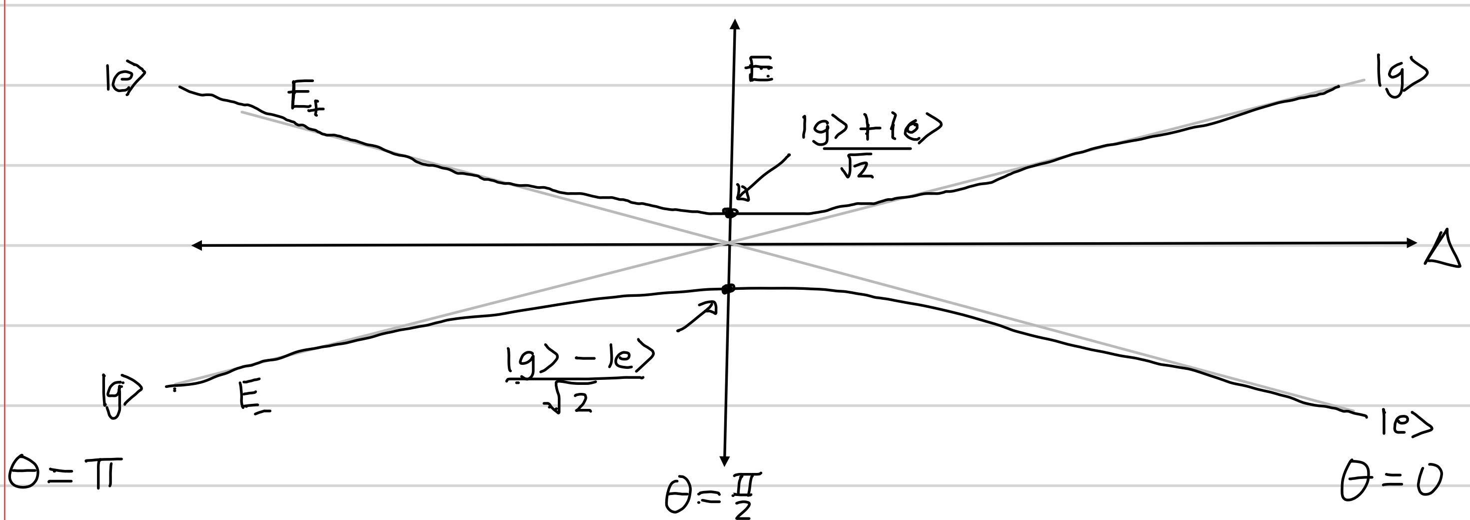
$$\Rightarrow \hat{H}_{RF}(t) = \hbar\Omega_{tot}\,\vec{e}_n(t)\cdot\hat{\vec{\sigma}}, \quad \Omega_{tot} = |\vec{\Omega}_{tot}(t)|, \quad \vec{e}_n(t) = \frac{\vec{\Omega}_{tot}(t)}{|\vec{\Omega}_{tot}(t)|} = \cos\theta(t)\vec{e}_z + \sin\theta(t)\vec{e}_x$$

$$\tan\theta(t) = \frac{\Omega_{tot}(t)}{-\Delta(t)}; \quad \cos\theta(t) = \frac{-\Delta(t)}{\Omega_{tot}(t)}, \quad \sin\theta(t) = \frac{\Omega}{\Omega_{tot}(t)}$$

We can immediately write the eigenvectors and eigenvalues of $\hat{H}_{RF}(t)$

$$E_\pm(t) = \pm\frac{\hbar\Omega_{tot}(t)}{2} = \pm\frac{\hbar}{2}\sqrt{\Delta^2(t)+\Omega^2}$$

$$|\pm(t)\rangle = |\updownarrow_{\vec{e}_n(t)}\rangle = \cos\left[\frac{\theta(t)}{2}\right]|g\rangle \pm \sin\left[\frac{\theta(t)}{2}\right]|e\rangle$$



According to the adiabatic theorem of quantum mechanics, for a time-dependent Hamiltonian, if at time $t=0$ $|\psi(0)\rangle = |u_n(0)\rangle$ where $\hat{H}(0)|u_n(0)\rangle = E_n(0)|u_n(0)\rangle$, then at a later time, $|\psi(t)\rangle \approx |u_n(t)\rangle$, where $\hat{H}(t)|u_n(t)\rangle = E_n(t)|u_n(t)\rangle$, if the evolution is "adiabatic". In other words, if we start in an eigenstate of the Hamiltonian, then we stay in the time-local eigenstate of the Hamiltonian if the Hamiltonian changes slowly enough. For the case at hand, the adiabatic eigenstate are $|\pm(t)\rangle$. These are sometimes known as the dressed states because the atomic levels are "dressed" by the laser field field. In contrast, the states $|g\rangle, |e\rangle$ are known as the "bare states."

The adiabatic theorem of quantum mechanics state that the evolution will remain adiabatic in the sense described above if

$$\frac{|\langle +(t)| \cdot \frac{d\hat{H}}{dt} |-(t)\rangle|}{(E_+(t) - E_-(t))^2/\hbar} \ll 1$$

The denominator is the "gap" between the energies of the dressed energy levels.

$$\hat{H}(t) = \hbar|\vec{\Omega}_{tot}(t)|\, \vec{e}_n(t)\cdot\hat{\vec{\sigma}} \qquad \vec{e}_n(t) = \cos\theta(t)\,\vec{e}_z + \sin\theta(t)\,\vec{e}_x$$

$$\Rightarrow \frac{d\hat{H}}{dt} = \hbar\dot{\Omega}_{tot}\,\vec{e}_n(t)\cdot\hat{\vec{\sigma}} + \hbar\Omega_{tot}\dot{\theta}\,[-\cos\theta\,\hat{\sigma}_z + \sin\theta\,\hat{\sigma}_x]$$

$$\Rightarrow \langle +(t)|\, \hbar\frac{d\hat{H}}{dt}\,|-(t)\rangle = \hbar^2\Omega_{tot}\dot{\theta}$$

$$\Rightarrow \text{Adiabatic if } \frac{|\hbar^2\Omega_{tot}\dot{\theta}|}{\Omega_{tot}^2} \ll 1 \quad \Rightarrow |\dot{\theta}| \ll \Omega_{tot} \geq \Omega \text{ (the minimum gap)}$$

$$\Rightarrow \boxed{\text{Require } |\dot{\theta}| \ll \Omega \text{ to be adiabatic}}$$

This is exactly the same condition we saw in the geometric picture on the Bloch sphere. In order to evolve adiabatically, the rate of precession of the Bloch vector, $\Omega_{tot} \geq \Omega$, must, at all times, remain fast compared to the rotation rate of the torque vector, $\dot{\theta}$. Simultaneously, we must have $\dot{\theta} \gg \Gamma$ to avoid decay.

Problem 3: Light Forces on Atoms

Given at monochromatic, polarized electric field at the position of an atom

$$\vec{E}(\vec{R},t) = \vec{\epsilon}_L \underbrace{E_0(\vec{R})}_{\text{position-dependent amplitude}} \cos(\omega_L t + \underbrace{\phi(\vec{R})}_{\text{position-dependent phase}})$$

The atom-light interaction Hamiltonian, in the rotating frame, RWA is

$$\hat{H}_{AL}(\vec{R}) = \frac{\hbar\Omega(\vec{R})}{2}\left[e^{-i\phi(\vec{R})}|e\rangle\langle g| + e^{i\phi(\vec{R})}|g\rangle\langle e|\right]$$

$$\text{where } \hbar\Omega(\vec{R}) = -\langle e|\vec{d}|g\rangle\cdot\vec{\epsilon}_L E_0(\vec{R})$$

The mean force on the atom $\vec{F} = -\langle\vec{\nabla}\hat{H}_{AL}(\vec{R})\rangle$

(a) $\vec{F} = -\mathrm{Tr}(\hat{\rho}\,\vec{\nabla}H_{AL}(\vec{R})) = -\frac{\hbar}{2}\left[\vec{\nabla}\Omega\left(e^{-i\phi}\rho_{ge} + e^{i\phi}\rho_{eg}\right) - i\Omega\vec{\nabla}\phi\left(e^{-i\phi}\rho_{eg} - e^{i\phi}\rho_{ge}\right)\right]$

$\phi(\vec{R}) = 0$ at the position of the atom

$\Rightarrow \vec{F} = -\hbar\left[\vec{\nabla}\Omega\,\mathrm{Re}(\rho_{eg}) + \Omega\vec{\nabla}\phi\,\mathrm{Im}(\rho_{eg})\right]$ $\qquad \rho_{eg} = \frac{1}{2}(u + iv)$

Bloch vector components

$$\Rightarrow \boxed{\vec{F} = \vec{F}_{diss} + \vec{F}_{react}, \text{ where } \vec{F}_{diss} = -\frac{\hbar}{2}v\,\nabla\Omega, \quad \vec{F}_{react} = -\frac{\hbar u}{2}\Omega\vec{\nabla}\phi}$$

(b) Following our studies of the classical Lorentz oscillator model, the rate at which the field does work on the atom is

$$\frac{dW}{dt} = \overbrace{\dot{\vec{d}}\cdot\vec{E}_L(\vec{R},t)}^{\text{time average}}, \quad \text{where } \dot{\vec{d}} = \frac{d}{dt}\overbrace{d_{eg}(u\cos\omega_L t - v\sin\omega_L t)}^{\text{mean atom dipole}} = -\omega_L d_{eg}(u\sin\omega_L t + v\cos\omega_L t)$$

$$\Rightarrow \frac{dW}{dt} = \overline{-\omega_L d_{eg}E_0(u\sin\omega_L t + v\cos\omega_L t)\cos\omega_L t} = +\hbar\omega_L\Omega_0 v/2 \quad (\text{where } \hbar\Omega_0 = -d_{eg}E_0)$$

In steady state: $\frac{dW}{dt} = +\frac{\hbar\Omega_0}{2}\omega_L v_{s.s.} = \frac{\hbar\Omega_0}{2}\omega_L\left(\frac{\Gamma}{\Omega_0}\frac{s}{1+s}\right)$

Having used the steady state solution to the Bloch equations

$$\boxed{\Rightarrow \frac{dW}{dt} = \hbar\omega_L\,\Gamma\underbrace{\frac{s/2}{1+s}}_{\text{steady-state population in excited state}} = \hbar\omega_L\,\Gamma\rho_{ee}^{s.s.} = \hbar\omega_L\,\gamma_s}$$

$\gamma_s = \Gamma\rho_{ee}^{s.s.}$ = probability to be in the excited state $\times$ rate of spontaneous emission
= photon scattering rate.

Interpretation: The (time averaged) rate at which the field does work on the atom is equal to the rate at which the atom scatters photons times energy $\hbar\omega_L$/photon. In other words, the every photon of the laser field absorbed by the atom does work on the atom. Photons are reëmitted in random directions, and thus do no work on the atoms.

(c) For a plane wave, $\phi(\vec{R}) = -\vec{k}_L\cdot\vec{R}$, where $\vec{k}_L$ is the laser beam's wave vector

From part (b), $\vec{F}_{diss} = -\frac{1}{2}\hbar v\Omega(\vec{R})\vec{\nabla}(-\vec{k}_L\cdot\vec{R}) = \hbar\vec{k}_L\frac{\Omega}{2}v \Rightarrow \boxed{\vec{F}_{diss} = \gamma_s\hbar\vec{k}_L}$

The dissipative force is also known as the "scattering force." Each scattered photon imparts a moment $\hbar k_L$ on the atom. The rate of photon impulses = γ_s = scattering rate.

(d) Consider the reactive force for weak saturation, $s \ll 1 \Rightarrow u \approx \frac{2\Delta}{\Omega}s = \frac{\Omega\Delta}{\Delta^2+\Gamma^2/4}$

$$\vec{F}_{react} = -\frac{1}{2}\hbar u\vec{\nabla}\Omega = -\frac{1}{2}\hbar\Delta\frac{\Omega\vec{\nabla}\Omega}{\Delta^2+\Gamma^2/4} = -\vec{\nabla}\left\{\hbar\Delta\frac{\Omega^2(\vec{R})/4}{\Delta^2+\Gamma^2/2}\right\} \Rightarrow \boxed{\vec{F}_{react} = -\vec{\nabla}U_{L.S.}(\vec{R})}$$

The light-shift potential $\boxed{U_{L.S.} = \frac{\hbar\Delta}{2}s(\vec{R}) = \frac{\hbar\Delta}{4}\frac{d_{ge}^2 E_0(\vec{R})^2/\hbar^2}{\Delta^2+\frac{\Gamma^2}{4}} = -\frac{1}{4}\mathrm{Re}(\tilde{\alpha}(\Delta))\,|E_0(\vec{R})|^2}$

Where $\tilde{\alpha}(\omega) = \frac{d_{eg}^2}{-\hbar(\Delta + i\frac{\Gamma}{2})}$.

U_{LS} is the conservative potential associated with a polarizable particle in an ac-electric field.